EXPLORING PROSPECTIVE MATHEMATICS TEACHERS’ MATHEMATICAL KNOWLEDGE FOR TEACHING ALGEBRAIC PROBLEM SOLVING

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ABSTRACT

The development of students’ problem solving abilities is a primary objective of mathematics instruction. A wide range of factors and decisions must be considered to reach this goal. The present intense interest in research on teachers’ knowledge and proficiencies demands that future problem-solving research pay close attention to the mathematical and pedagogical knowledge and proficiencies a teacher should possess. Thus, this study explored the prospective mathematics teachers’ mathematical knowledge for teaching algebraic problem solving. The study was conducted with 10 prospective teachers who were undergoing their practical teaching in several secondary schools. The data were obtained via task-based open questionnaire. The data obtained were analysed in accordance with the content analysis by focusing on the issues related to mathematical knowledge for teaching highlighted in the literature. The findings indicated that majority of the prospective teachers were not familiar with algebraic problem-solving methods (e.g., heuristics, strategies) that are accessible to students. They normally used the four step-by-step Polya Method in teaching mathematical problems. It suggests the need for prospective teachers to reflect on their learning experience from the perspective as learner and as teacher to acquire highly proficient teachers in constructing a meaningful instructional strategy for algebraic problem solving.

Keywords: Algebraic problem solving, secondary mathematics teaching, mathematical pedagogical knowledge.

1. Introduction

Mathematical problem solving is emerging to be a key concern in current reform and debate in Malaysian educational setting and elsewhere (Lester, 2013). In fact, this area of study has received a great deal of attention for a considerable period. This is largely due to the fact that mathematical problem solving provides a window through which we can view how students engage in highly complex cognitive activities involving doing routine and non-routine word problems, creating patterns, interpreting figures, making conjectures reasoning and proof, etc.

Most mathematics educators consent that the development of students’ problem solving abilities is a primary objective of instruction and how this goal is to be reached involves consideration by the teacher of a wide range of factors and decisions. In this study, we focus on ideas associated with this complex and elusive area of study by giving special attention to algebraic problem-solving instruction, an area that has not received a great of attention (Lester, 2013). Specifically, this study is related to teachers’ knowledge which concerned with the teacher’s thinking while facilitating students’ attempts to understand a task, develop a plan for solving it, carry out the plan to obtain an answer, and look back over the solution effort in teaching algebraic problem solving.

Mathematical Problem Solving: Malaysian Context

Findings of international studies such as Trends in Mathematics and Science Study (TIMSS) (Mullis et al. (2012) and Programme of International Student Assessment (PISA) (Organisation for Economic Co-operation and Development (OECD), 2013) showed that most high school students in Malaysia continue to perform at less than satisfactory levels. They have not reached a satisfactory level of higher order thinking, specifically in mathematical problems solving. Towards that end, Malaysian National Education Transformation Plan 2013-2025 (Ministry of Education, 2013) has emphasised the need to develop quality mathematics teachers for the future who are capable of making innovative and creative pedagogical decisions in varying contexts.

For example, in the analyses of Malaysian students’ performance in TIMSS in a number of years, Mullis, Martin, Foy & Arora (2012) found that only 2-10% of the students are capable of interpreting the information, drawing conclusions and generalization in solving complex problems – activities that collectively reflect low levels of activation of cognitive processses in mathematical problem solving. Mullis and colleagues also showed that 60% of students achieved the low international benchmark. These results suggest that the students understand the basic mathematical concepts but, in general, they are not able to transfer that knowledge to non-routine problem situations (Ministry of Education, 2013). Likewise, another international study, namely, the PISA 2009 showed that Malaysia students’ mathematical performances were located in the bottom one third of all the 74 participating countries (Walker, 2011).

As in the TIMSS study, PISA’s report for mathematics achievement showed that only a small proportion (8%) of Malaysian students achieved advanced levels of thinking. Overall, trends in TIMSS and PISA provide evidence of Malaysian students’ continuing difficulty in solving mathematical tasks which involve complex interpretation and synthesis – key aspects of mathematical problem solving. The solution of complex mathematical problems involves the transfer of prior learning to new contexts, and this transfer, we argue, can be facilitated by the acquisition of appropriate mathematical problem solving instruction.

Algebraic Problem Solving

The field of algebra progressively has become a central theme throughout mathematics education, with substantial literature discussing the development of algebraic reasoning in elementary and middle grades (Blanton & Kaput, 2003; Carraher & Schliemann, 2007, Blair & Rich, 2011). Scholars highlighted the algebraic nature of arithmetic thinking and conceptualize the study of algebra as much more than just techniques to “solve for x.” Kaput (2008), for example, delineated algebra as: 1) the study of structures and systems abstracted from computations and relations; 2) the study of functions and joint variation; and 3) the application of modeling languages in and out of mathematics. Kaput’s (2008) and others’ (e.g., Usiskin, 1988) contributions broaden the scope of algebraic content to incorporate arithmetic ideas that help vertically align the development of algebraic reasoning throughout students’ learning. In this article, we specifically focus on only the secondary algebraic problem solving.

What is algebraic problem solving? Lester and Kehle (2003, p. 510) emphasised that:
Successful problem solving involves coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuition in an effort to generate new representations and related patterns of inference that resolve some tension or ambiguity (i.e., lack of meaningful representations and supporting inferential moves) that prompted the original problem-solving activity.

The above description of problem solving lies in its identification of several key ingredients of success: coordination of experience, knowledge, familiar representations, patterns of inference, and intuition (Lester, 2013). In other words, an individual must have ample relevant experience in learning how to solve problems, strong content knowledge, proficiency in using a variety of representations and a solid grasp of how to recognize and construct patterns of inference. Using Lester’s definition of general mathematical problem solving, algebraic problems solving can be referred as an activity requiring an individual to engage in a variety of algebraic cognitive processes namely the coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuition. Each of these key points are helpful for thinking about how to teach students to solve problems or equipping prospective teachers with the proficiencies needed to teach mathematics either for or via problem solving.

Problems in Teaching and Learning Algebraic Problem Solving

Research on problem solving suggests that many students tend to give up rather quickly when presented with novel or unfamiliar problem solving tasks (Doerr, 2006). Accordingly, numerous researchers have studied the obstacles that students meet when they learn algebra (Johanning, 2004; Malisani & Spagnolo, 2009; Egodowatte, 2011). For example, a study by Egodowatte (2011) reported that students made errors and misconceptions related to the concept of variables, algebraic expressions, algebraic equations and word problems. This was caused by teachers always emphasized on computational procedures and made algebraic manipulation of symbols (Johanning, 2004). This traditional algebraic learning does not provide thoughts and give meaning to the learning activities.

Another experimental study by Malisani & Spagnolo (2009) added that the students meet serious difficulties in translating from algebraic language (6x–3y = 18) into natural language. Only 6 students (out of all 115participants in the first and second experiments) correctly formulate a problem in a different context from the “money and bets” of the first question. They observe an important rupture between symbolic language and the possibility of finding a context to give meaning to the equation. They argued that this behaviour is the result of insufficient control of the symbols and it may also represent the impossibility of interpreting a linear equation in functional relation terms is translated into the impossibility of identifying a function in a situation/problem.

Confusion about the function of variables also can contributed to the difficulties in the process of translating problem situations into appropriate algebraic expressions (Bishop, Filloy & Puig, 2008; Egodowatte; 2011). These processes required students to identify the variables, constants, and find the relationships that are formed. The aspect of relationship between variables in algebraic problems solving situations creates difficulty when students try to change to form symbols. In addition, Capcaro (2006) found that most students translate algebraic word problem solving into mathematical expressions from left to right. For example, for “Three less than a number” , most students translate it to 3 - x based on word less than ( refer to less ), while the correct answer is x - 3 where x refers to a number. This misconception needs to be emphasized by teachers in the mathematics classroom so that students are aware of their understanding of transforming word problems into mathematical expressions.
According to Martinez (2002), even though some of the algebraic thinking and important concepts of algebra have been taught at early grade levels, but there are still many students assume that learning algebra is elusive. In the TIMSS study, he found that American students apparently success in answering questions involving procedural skills and routine problems, such as finding the value of x in a linear equation. However, they have difficulty answering questions that require understanding and use of algebraic expressions to perform complex procedures. Martinez found problems faced by students in solving algebraic problems were problems which require complex procedures that needed mastering concept of equivalence of algebraic expressions.

In sum, review of literature showed that there were many reasons in learning algebra and algebraic problem solving. Rather than blaming students’ ability and weaknesses, we should see in different view such that to how teachers teach algebraic problem solving might impact students’ learning. Therefore, a major concern of mathematics educators is the efforts to improve teaching algebraic problem solving so that students will be more engaged and be more effective problem solvers. Although research on mathematical problem solving has provided some valuable information about problem-solving instruction, however Lester (2013) argued that we haven’t learned nearly enough. Thus, the present study is grounded in the above issues by exploring the prospective mathematics teachers’ mathematical knowledge for teaching algebraic problem solving.

2. Theoretical Framework

Theoretical framework for this study are based on concepts of mathematical knowledge for teaching by Ball, Bass and Hill (Ball & Bass, 2003, Hill, Ball, & Shilling, 2008; Ball, Thames, & Phelps, 2008) and model of complex mathematical activity by Lester’s (2013) to guide the analysis and interpretation of results. However, as part of a larger study, this paper will only focus on concepts of mathematical knowledge for teaching.

Mathematical Knowledge for Teaching

Mathematical knowledge for teaching [MKT] is gained by observing and cataloging the specific knowledge required by classroom teachers to perform their jobs; it refers to a specific body of knowledge distinct from mathematical content knowledge that teachers must know and understand to enable them to teach effectively. Several studies suggest that the nature, depth, and organization of teacher knowledge influences teachers’ presentation of ideas, flexibility in responding to students’ questions, and capacity for helping students connect mathematical ideas (e.g., Ball, 1988; Stein, Baxter, & Leinhardt, 1990). Mathematics teachers are a special class of users of mathematics; the knowledge they need to teach mathematics goes beyond what is needed by other well-educated adults, including mathematicians (Ball, Lubienski & Mewborn, 2001; Ball, Bass & Hill, 2004; Ball, Thames, & Phelps, 2008). As described by Hill, Rowan and Ball (2005):

Mathematical knowledge for teaching goes beyond that captured in measures of mathematics courses taken or basic mathematical skills. For example, teachers of mathematics not only need to calculate correctly but also need to know how to use pictures or diagrams to represent mathematics concepts and procedures to students, provide students with explanations for common rules and mathematical procedures, and analyze students’ solutions and explanations (p. 372).

Further, Hill and colleagues associate higher levels of MKT to greater student learning gains, underscoring the importance of teachers’ MKT and its impact on what students learn in the classroom (Hill et al., 2005).
Figure 2 depicted the Model of MKT (Ball & Bass, 2003, Hill, Ball, & Shilling, 2008; Ball, Thames, & Phelps, 2008). Theoretically, the MKT construct follows Shulman’s (1986) efforts to define the theories concerning subject matter knowledge (SMK) and pedagogical content knowledge (PCK). A proposed strand of the MKT composed of each of the six portions of the oval. The left side of the oval, labelled Subject Matter Knowledge contains two strands: Common Content Knowledge (CCK) and Specialized Content Knowledge (SCK). CCK refers as to knowledge that is used in the work of teaching in ways in common with how it is used in many other professions, that also use mathematics. Whilst, SCK allows teachers to engage in particular teaching tasks such as how to represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems (Ball et al., 2005). The right side of the oval represents strands associated with Shulman’s proposed PCK, and contained KCS, Knowledge of Content and Teaching (KCT) and Knowledge of Curriculum.

Ball, Bass, and Hill (2008) categorized mathematical knowledge for teaching into four domains: (a) common content knowledge (CCK), (b) specialized content knowledge (SCK), (c) knowledge of students and content (KSC), and (d) knowledge of teaching and content (KTC). To illustrate these four domains, consider the difference between calculating the answer to a multi-digit multiplication problem (CCK), analyzing calculation errors for the problem (SCK), identifying student thinking that is likely to have produced such errors (KSC), and recognizing which manipulatives would best highlight place-value features of the algorithm (KTC) (Ball, Sleep & Thames, 2007, p. 4).

The present investigation blended on an approach involving creating conceptual framework for MKT by mathematical topic, drawing on research on the teaching and learning of that topic as suggested by Stylianides & Ball (2004). Specifically, this study involved examining teachers’ MKT related to
algebraic problem solving strategy, a content area that has historically posed challenges both for students and teachers, yet is fundamental to mathematics. Using a model grounded on teaching and learning of mathematical problem solving, MKT was assessed through written instrument.

**Model of Complex Mathematical Activity**

Model of complex mathematical activity (Lester, 2013) provides a much broader framework of mathematical activity during problem solving and gives a prominent role to the metacognitive activity engaged in by the individual or group. Figure 2 depicts the model which encompasses a complex mathematical behavior, involved, multiphase processes that begins when an individual working in a complex context, poses (or is given) a specific task to solve which resulted in a realistic representation of the original situation, abstraction phase, manipulating the mathematical representation and deducing some mathematical conclusions phase, and finally comparing the conclusions/results obtained with the original context and problem phase. The details elaboration of each phase can be referred in Lester (2013). This concept was used to guide the analysis and interpretation of results in this study.

![Figure 2: A model of complex mathematical activity](image)


**Purpose of the Study**

We have argued that solving algebraic problems is a complex process that is influenced by various factors including teachers’ instruction. The present intense interest in research on teachers’ knowledge and proficiencies demands that future problem-solving research pay close attention to the mathematical and pedagogical knowledge and proficiencies a teacher should possess. Specifically, the teacher planning has been given too little attention as a factor of importance in problem-solving instruction research (Lester, 2013). Accordingly, the aim of the study is to explore the prospective mathematics teachers’ mathematical knowledge for teaching algebraic problem solving. In this study, one particular type of mathematical knowledge for teaching will be considered...
namely strategy for teaching algebraic problem solving. We sought data relevant to the following two research questions:

1. What are the strategies suggest by the prospective mathematics teachers in teaching algebraic problem solving?
2. What is the prospective mathematics teachers’ mathematical knowledge for teaching algebraic problem solving?

3. Methodology

Design

The case study model, which is among the qualitative, naturalistic research perspective (Creswell, 2008) was used in this study which focus on capturing and interpreting participants’ thinking about a phenomenon, problem solving strategy in this case.

Participants

The study was conducted with 40 prospective mathematics teachers [PMTs] who were undergoing their practical teaching in secondary schools, Malaysia, in the seventh semester of their 4-year degree Mathematics Education Programme. However, in this paper, only 10 of their instruments were randomly selected and discussed they had taken all mathematics education courses and had completed all of their mathematics required for the programme. Thus, they had instruction or theory on mathematical problem solving as well as their content mathematics knowledge.

Instrumentation

The data were obtained via task-based questionnaire. PMTs were required to respond on one question related to the strategy of teaching algebraic problem solving. The question is as follows:
Describe in detail, the strategies that you will use to teach form four students to solve the following algebraic problem. Attached is a schematic solution of this question as a guide. If the space is insufficient, please use the provided papers.

In the above diagram, ABCD is a piece of rectangular paper with an area of 28 cm². AEB is a semicircle-shaped cut from the paper. The remaining perimeter of the paper is 26 cm. Find the integer values of x and y (Use $\pi = \frac{22}{7}$).

Figure 2: Question related to the strategy of teaching algebraic problem solving

4. Data Analysis
The data obtained were analysed in accordance with the content analysis by focusing on the issues related to mathematical knowledge for teaching and complex mathematical behaviour during solving mathematical problem as highlighted in the literature. The analysis began with open-ended coding (Strauss & Corbin, 1998) of the data. Coding involved, for example, identifying significant statement about PMTs thinking about the strategy or problem solving instruction. The coded information was categorised based on common themes and frequency of occurrence.

5. Results and Discussion
The results are presented in terms (i) strategies suggest by the PMTs in teaching algebraic problem solving and (ii) PMTs’ mathematical knowledge for teaching algebraic problem solving.

Strategy in Teaching Algebraic Problem Solving

There was variety in the nature of PMTs’ strategy in teaching algebraic problem solving. There were seven categories that emerged as their dominant ways of thinking strategy in teaching how to solve the problem. Table 1 presented those categories with correspond frequency occurrences. Some of the PMTs used more than one strategy in teaching the problem. The findings indicated that 50% of the PMTs normally used the four step-by-step Polya Method in teaching the algebraic problem solving. Secondly, 40% of them are comfortable using explanation method. This indicated that they were not familiar with algebraic problem-solving methods (e.g., heuristics, strategies) that are accessible to students. 30% of the PMTs using concrete materials such as the manila card and A4 paper and paper cutting strategy in explaining the question asked and understanding how to create two simultaneous equations to solve for x and y. 20% of them integrated the questioning technique in order to elicit students’ ideas, understanding of the question and to prompt them. 10% were using the drawing a diagram strategy and the practical method in teaching the algebraic problem solving.
Table 1: Category of PMTs’ strategy in teaching algebraic problem solving

<table>
<thead>
<tr>
<th>Category</th>
<th>PMT</th>
<th>Frequency (Percentage)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Four step-by-step Polya method</td>
<td>3, 4, 5, 9, 10</td>
<td>5 (50%)</td>
</tr>
<tr>
<td>Exposition</td>
<td>2, 6, 7, 8</td>
<td>4 (40%)</td>
</tr>
<tr>
<td>Paper cutting</td>
<td>1, 3, 7</td>
<td>3 (30%)</td>
</tr>
<tr>
<td>Using concrete material</td>
<td>1, 7, 8</td>
<td>3 (30%)</td>
</tr>
<tr>
<td>Questioning technique</td>
<td>8, 9</td>
<td>2 (20%)</td>
</tr>
<tr>
<td>Drawing a diagram</td>
<td>6</td>
<td>1 (10%)</td>
</tr>
<tr>
<td>Practical method</td>
<td>1</td>
<td>1 (10%)</td>
</tr>
</tbody>
</table>

Mathematical Knowledge for Teaching Algebraic Problem Solving

In terms of MKT Algebraic Problem Solving, all 10 PMTs exhibited the CCK, 7 of them exhibited the KSC and only 1 exhibited the SCK. Table 2 displayed the types of MKT exhibited by the PMTs.

Table 2: Types of MKT exhibited by the PMTs

<table>
<thead>
<tr>
<th>Prospective Mathematics Teacher (PMT)</th>
<th>Mathematical Knowledge for Teaching Algebraic Problem Solving</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>• Procedural and conceptual knowledge of solving the problem (CCK)</td>
</tr>
<tr>
<td></td>
<td>• Identifying students’ basic knowledge of area of rectangle, perimeter of irregular shape and basic algebra (KSC)</td>
</tr>
<tr>
<td>2</td>
<td>• Procedural and conceptual knowledge of solving the problem (CCK)</td>
</tr>
<tr>
<td></td>
<td>• Identifying students’ basic knowledge of area or rectangle, perimeter of semi-circle and basic algebra (KSC)</td>
</tr>
<tr>
<td>3</td>
<td>• Procedural and conceptual knowledge of solving the problem (CCK)</td>
</tr>
<tr>
<td>4</td>
<td>• Procedural and conceptual knowledge of solving the problem (CCK)</td>
</tr>
<tr>
<td></td>
<td>• Identifying students’ basic knowledge of area of rectangle and perimeter of semi-circle (KSC)</td>
</tr>
<tr>
<td>5</td>
<td>• Procedural and conceptual knowledge of solving the problem (CCK)</td>
</tr>
<tr>
<td></td>
<td>• Analysing calculation errors for the solution of the problem (SCK)</td>
</tr>
<tr>
<td></td>
<td>• Identifying students’ basic knowledge of area of rectangle, perimeter, circumference of circle, simultaneous equation and factorization (KSC)</td>
</tr>
<tr>
<td>6</td>
<td>• Procedural and conceptual knowledge of solving the problem (CCK)</td>
</tr>
<tr>
<td>7</td>
<td>• Procedural and conceptual knowledge of solving the problem (CCK)</td>
</tr>
<tr>
<td>8</td>
<td>• Procedural and conceptual knowledge of solving the problem (CCK)</td>
</tr>
<tr>
<td>9</td>
<td>• Procedural and conceptual knowledge of solving the problem (CCK)</td>
</tr>
</tbody>
</table>
Identifying students’ basic knowledge of area of rectangle, perimeter of irregular shapes and basic algebra (KSC)

Procedural and conceptual knowledge of solving the problem (CCK)

Identifying students’ basic knowledge of area, perimeter and basic algebra (KSC)

All the PMTs have the procedural and conceptual knowledge of the algebraic problem solving that was asked in the question. The PMTs demonstrated their understanding of a method to accomplish the task. The seven categories that they have used in teaching the task, all lead to the correct solution. For example, PMT 5 exhibited the CCK by using the four step-by-step method by firstly asking students to extract the information given in the diagram. Then, PMT 5 guided the students to plan a strategy such as finding two equations using the information given involving variables x and y and finally solve the simultaneous equation. Next, PMT 5 guided students to perform those strategies. Finally, PMT 5 made sure that the students checked their answers by substituting both sets of answers into one of the equations such that the correct solution must satisfy the equation. The same strategy was also employed by PMTs 3, 4, 9 and 10. However, only PMT 5 exhibited the SCK where she had managed to analyze calculation errors for the solution of the problem. The finding indicated that the SCK seemed to be ignored by most of the PMTs. This SCK is very important as it refers to a specific body of knowledge distinct from mathematical content knowledge that teachers must know and understand to enable them to teach effectively (Hill, Rowan & Ball, 2005; Holmes, 2012).

Other examples are PMT 2, 6, 7 and 8 who were exhibited the CCK by using the exposition method where they just demonstrated to students on how to work out the problem and students will follow their instructions passively. PMT 1 employed a combination strategy of practical method, using concrete material and paper cutting to teach the task to students. One important point that had been noted in this finding was that the PMTs did not recognised or familiar with the best strategy in teaching the task which could demonstrate their KTC. This could be further explored by interviewing each of PMTs to investigate more about their KTC and the reasons they had choose those strategies.

In addition, most of the PMTs (60%) showed their KSC knowledge accept PMT 3, 6, 7 and 8. For example, PMT 5 identified students’ basic knowledge of area of rectangle, perimeter, circumference of circle, simultaneous equation and factorization. PMT 1 also identified their students’ basic knowledge of area of rectangle, perimeter of irregular shape and basic algebra.

**Conclusion and Implication**

In summary, the findings indicated that majority of the prospective teachers were not familiar with the best algebraic problem-solving methods (e.g., heuristics, strategies) that are accessible to students to teach the task. They normally used the four step-by-step Polya Method in teaching mathematical problems. They were also comfortable of using the exposition method by explaining how to work out on the task.

The above findings could be attributed to a number of factors. Firstly, it is possible that PMTs were not given explicit instructions about the various algebraic instructional stragies and the role of the strategies in solving complex algebraic problems. Secondly, the lecturers may not understand the nature of MKT algebraic problem solving and its role in helping students to solve non-routine algebraic problem solving. Based on the above assumptions, we conclude that teachers’ professional programme in Malaysia must provide explicit instruction in MKT algebraic problem solving. In addition, the SCK should be emphasized explicitly as PMTs must know and understand it to enable them to teach effectively.
Regular mathematics education courses should allocate time to support PMTs to reason without any constraints to produce correct or incorrect answers to predetermined outcomes. The current reform initiated by the Malaysian government in promoting higher order thinking skills is not grounded in a complete understanding of what these skills are and how they are played out in algebraic problem solving. Further research should be conducted to generate higher level of clarity about the roles of algebraic thinking in mathematics learning.

Finally, it suggests the need for PMTs to reflect on their learning experience from the perspective as learner and as teacher to acquire highly proficient teachers in constructing a meaningful instructional strategy for algebraic problem solving.

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